A Complete Discrimination System for Polynomials*

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Given a polynomial with symbolic/literal coefficients, a complete discrimination system is a set of explicit expressions in terms of the coefficients, which is sufficient for determining the numbers and multiplicities of the real and imaginary roots. Though it is of great significance, such a criterion for root-classification had never been given for polynomials with degrees greater than 4. The lack of efficient tools in this aspect extremely prevents computer implementations for Tarski's method and other ones in Automated Theorem Proving. To remedy this defect, the authors propose a generic algorithm to produce a complete discrimination system for a polynomial with any degree. This result has extensive applications in various fields, and its efficiency was demonstrated by computer implementations.

For example, the following table gives the numbers of real and imaginary roots and multiplicities of repeated roots of polynomial $g_5 = x^5 + px^3 + qx^2 + rx + s$ in all cases:

(1)	$D_5 > 0 \wedge D_4 > 0 \wedge D_3 > 0 \wedge D_2 > 0$	$\{1,1,1,1,1\}$
(2)	$D_5 > 0 \wedge (D_4 \le 0 \vee D_3 \le 0 \vee D_2 \le 0)$	$\{1\}$
(3)	$D_5 < 0$	$\{1,1,1\}$
(4)	$D_5 = 0 \wedge D_4 > 0$	$\{2,1,1,1\}$
(5)	$D_5 = 0 \wedge D_4 < 0$	$\{2,1\}$
(6)	$D_5 = 0 \wedge D_4 = 0 \wedge D_3 > 0 \wedge E_2 \ne 0$	$\{2,2,1\}$
(7)	$D_5 = 0 \wedge D_4 = 0 \wedge D_3 > 0 \wedge E_2 = 0$	$\{3,1,1\}$
(8)	$D_5 = 0 \wedge D_4 = 0 \wedge D_3 < 0 \wedge E_2 \ne 0$	$\{1\}$
(9)	$D_5 = 0 \wedge D_4 = 0 \wedge D_3 < 0 \wedge E_2 = 0$	$\{3\}$
(10)	$D_5 = 0 \wedge D_4 = 0 \wedge D_3 = 0 \wedge D_2 \ne 0 \wedge F_2 \ne 0$	$\{3,2\}$
(11)	$D_5 = 0 \wedge D_4 = 0 \wedge D_3 = 0 \wedge D_2 \ne 0 \wedge F_2 = 0$	$\{4,1\}$
(12)	$D_5 = 0 \wedge D_4 = 0 \wedge D_3 = 0 \wedge D_2 = 0$	$\{5\}$

*This work is supported in part by the National Science Foundation of China. Participation in and presentation of the paper at the conference was made possible through a generous grant from UNU/IIST, the United Nations University. International Institute for Software Technology, Macau.

where

$$
\begin{aligned}
D_2 &= -p \\
D_3 &= 40rp - 12p^3 - 45q^2 \\
D_4 &= 12p^4r - 4p^3q^2 + 117prq^2 - 88r^2p^2 - 40qp^2s + 125ps^2 \\
&\quad -27q^4 - 300qrs + 160r^3 \\
D_5 &= -1600qsr^3 - 3750ps^3q + 2000ps^2r^2 - 4p^3q^2r^2 + 16p^3q^3s \\
&\quad -900rs^2p^3 + 825q^2p^2s^2 + 144pq^2r^3 + 2250q^2rs^2 \\
&\quad +16p^4r^3 + 108p^5s^2 - 128r^4p^2 - 27q^4r^2 + 108q^5s \\
&\quad +256r^5 + 3125s^4 - 72p^4rsq + 560r^2p^2sq - 630prq^3s \\
E_2 &= 160r^2p^3 + 900q^2r^2 - 48rp^5 + 60q^2p^2r + 1500rpsq \\
&\quad +16q^2p^4 - 1100qp^3s + 625s^2p^2 - 3375q^3s \\
F_2 &= 3q^2 - 8rp
\end{aligned}
$$

These 6 polynomials form a discrimination system which is sufficient for the root classification of the quintic polynomial above.

The right column of the table describes the situations of the roots. For example, $\{1,1,1,1,1\}$ means 5 real simple roots and $\{2,2,1\}$ means 2 real double roots plus one real simple root. It is necessary to point out that there is only one real simple root in both cases (2) and (8), however, no repeated roots in case (2) while 2 imaginary double roots in case (8).

The result mentioned above is only a special case of the general criterion we established in this paper.