Gröbner Bases for Ideals in Monomial Algebras and their Application to Systems of Difference Equations

Franz PAUER and Andreas UNTERKIRCHER
Institut für Mathematik, Universität Innsbruck,
Technikerstraße 25, A-6020 Innsbruck, Austria
e-mail: Franz.Pauer@uibk.ac.at

Abstract

In [O] Gröbner basis techniques have been applied to systems of linear partial difference equations on $\mathbb{N}^n$. In [Z] and in [ZO] this method was extended to difference equations on $\mathbb{Z}^n$ by computing Gröbner bases for certain ideals in the polynomial ring in $2n$ variables. The aim of this paper is to present a more direct approach to difference equations on $\mathbb{Z}^n$ (and on finitely generated submonoids of it). We develop a basic theory of Gröbner bases for finite dimensional free modules over the algebra of Laurent polynomials $R[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ (and, more generally, over monomial subalgebras of it). We use and extend an idea of S. Zampieri (see [PZ]), who introduced generalized term orders and Gröbner bases with respect to them for ideals in polynomial rings (over coefficient fields). A generalized term order on a submonoid T of $\{x^i \mid i \in \mathbb{Z}^n\}$ is given by a conic decomposition $(T_i)_{i\in I}$ of T (i.e. a finite family of submonoids of T with $T \subseteq T_i \cdot T_i^{-1}, T_i^{-1} \cap T_i = \{1\}$ and $\bigcup_{i\in I} T_i = T$) and a total order $<$ on T such that 1 is the smallest element and for all $i \in I$, $r \in T$, $s, t \in T_i$ we have: $r < s$ implies $tr < ts$. For the sake of completeness we do not restrict ourselves to the case of coefficient fields, but admit coefficients in a commutative noetherian ring. In the last section we discuss several examples, among them the examples given in [Z] and [ZO]. For the latter our method yields the results without essential computations.

[O] U. Oberst. Multidimensional constant linear systems. Acta Applicandae Math., 20:1–175, 1990.

[PZ] F. Pauer and S. Zampieri. Gröbner Bases with respect to Generalized Term Orders and their Application to the Modelling Problem. Preprint 1995.

[Z] S. Zampieri. A Solution of the Cauchy Problem for Multidimensional Discrete Linear Shift-Invariant Systems. Linear Algebra and its Appl. 202:143-162, 1994.

[ZO] E. Zerz and U. Oberst. The Canonical Cauchy Problem for Linear Systems of Partial Difference Equations with Constant Coefficients over the Complete r-Dimensional Integral Lattice $\mathbb{Z}^r$. Acta Applicandae Math. 31: 249-273, 1993.