# Epsilon Functions

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# Descriptions

## epsilon

E1. `CharSet`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a set $\mathtt{P}$ of polynomials in $\mathtt{X}$, $\mathtt{CharSet}(\mathtt{P}, \mathtt{X})$ returns an $\mathbb{F}$-*modified quasi-characteristic set* [3] $\mathbb{C}$ of $\mathtt{P}$ with respect to the variable ordering $x_1 \prec \cdots \prec x_n$.

E2. `ICS`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set $\mathtt{P}$ or system $[\mathtt{P}, \mathtt{Q}]$ in $\mathtt{X}$, $\mathtt{ICS}(\mathtt{P}, \mathtt{X})$ or $\mathtt{ICS}([\mathtt{P}, \mathtt{Q}], \mathtt{X})$ returns an *irreducible characteristic series* [10] $\mathbb{T}_1, \ldots, \mathbb{T}_e$ of $\mathtt{P}$ or $[\mathtt{P}, \mathtt{Q}]$ with respect to the ordering $x_1 \prec \cdots \prec x_n$.

E3. `IVD`

Inputted a set or a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set $\mathtt{P}$ in $\mathtt{X}$, $\mathtt{IVD}(\mathtt{P}, \mathtt{X})$ returns a sequence of irredundant polynomial sets $\mathbb{P}_1, \ldots, \mathbb{P}_e$ such that

$$\mathrm{Zero}(\mathtt{P}) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{P}_i) \quad (1)$$

and each $\mathrm{Zero}(\mathbb{P}_i)$ as an algebraic variety is *irreducible*.[11]

Except in the case $e = 1$,

$$\sqrt{\mathrm{Ideal}(\mathtt{P})} = \bigcap_{i=1}^{e} \mathrm{Ideal}(\mathbb{P}_i)$$

also holds, where each $\mathrm{Ideal}(\mathbb{P}_i)$ is a prime ideal. The case $e = 1$ is different because the input $\mathtt{P}$ is simply returned when the variety $\mathrm{Zero}(\mathtt{P})$ is determined to be irreducible.

The computed $\mathbb{P}_i$, except in the case $e = 1$, are all Gröbner bases with respect to the purely lexicographical term order determined by $x_1 \prec \cdots \prec x_n$.

E4. `PID`

Inputted a set or a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set $\mathtt{P}$ in $\mathtt{X}$, $\mathtt{PID}(\mathtt{P}, \mathtt{X})$ returns a sequence of pairs $[\mathbb{P}_1, \mathbb{F}_1], \ldots, [\mathbb{P}_e, \mathbb{F}_e]$ of polynomial sets such that

$$\mathrm{Ideal}(\mathtt{P}) = \bigcap_{i=1}^{e} \mathrm{Ideal}(\mathbb{P}_i) \quad (2)$$

and each $\mathrm{Ideal}(\mathbb{P}_i)$ is a *primary ideal*[12] with $\mathrm{Ideal}(\mathbb{F}_i)$ as its associated prime.

The computed $\mathbb{P}_i$ are all Gröbner bases with respect to the purely lexicographical term order determined by $x_1 \prec \cdots \prec x_n$.

E5. `Prove`

Inputted the specification $\mathtt{T}$ of a geometric theorem, $\mathtt{Prove}(\mathtt{T})$ proves or disproves $\mathtt{T}$, with subsidiary conditions provided.

E6. `RegSer`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set $\mathtt{P}$ or system $[\mathtt{P}, \mathtt{Q}]$ in $\mathtt{X}$, $\mathtt{RegSer}(\mathtt{P}, \mathtt{X})$ or $\mathtt{RegSer}([\mathtt{P}, \mathtt{Q}], \mathtt{X})$ returns a *regular series*[7] $[\mathbb{T}_1, \mathbb{U}_1], \ldots, [\mathbb{T}_e, \mathbb{U}_e]$ of $\mathtt{P}$ or $[\mathtt{P}, \mathtt{Q}]$ under the variable ordering $x_1 \prec \cdots \prec x_n$.

E7. `RIM`

Inputted a list or a set $\mathtt{X} = \{x_1, \ldots, x_n\}$ of variables, a polynomial set $\mathtt{P}$ and a polynomial $\mathtt{p}$ in $\mathtt{X}$, $\mathtt{RIM}(\mathtt{P}, \mathtt{p}, \mathtt{X})$ returns `true` if $\mathtt{p} \in \sqrt{\mathrm{Ideal}(\mathtt{P})}$, or `false` otherwise.

E8. `SimSer`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set $\mathtt{P}$ or system $[\mathtt{P}, \mathtt{Q}]$ in $\mathtt{X}$, $\mathtt{SimSer}(\mathtt{P}, \mathtt{X})$ or $\mathtt{SimSer}([\mathtt{P}, \mathtt{Q}], \mathtt{X})$ returns a *simple series*[8] $[\mathbb{T}_1, \tilde{\mathbb{T}}_1], \ldots, [\mathbb{T}_e, \tilde{\mathbb{T}}_e]$ of $\mathtt{P}$ or $[\mathtt{P}, \mathtt{Q}]$ with respect to the variable ordering $x_1 \prec \cdots \prec x_n$.

E9. `TriSer`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set $\mathtt{P}$ or system $[\mathtt{P}, \mathtt{Q}]$ in $\mathtt{X}$, $\mathtt{TriSer}(\mathtt{P}, \mathtt{X})$ or $\mathtt{TriSer}([\mathtt{P}, \mathtt{Q}], \mathtt{X})$ returns a *fine triangular series* [4] $[\mathbb{T}_1, \mathbb{U}_1], \ldots, [\mathbb{T}_e, \mathbb{U}_e]$ of $\mathtt{P}$ or $[\mathtt{P}, \mathtt{Q}]$ under the variable ordering $x_1 \prec \cdots \prec x_n$.

E10. `Tsolve`

Inputted a list or a set $\mathtt{X} = \{x_1, \ldots, x_n\}$ of variables and a polynomial set $\mathtt{P}$ or system $[\mathtt{P}, \mathtt{Q}]$ in $\mathtt{X}$, $\mathtt{Tsolve}(\mathtt{P}, \mathtt{X})$ or $\mathtt{Tsolve}([\mathtt{P}, \mathtt{Q}], \mathtt{X})$ returns (all) the solutions of $\mathtt{P} = 0$ or $\mathtt{P} = 0, \mathtt{Q} \neq 0$ for $x_1, \ldots, x_n$.

Note that, for any polynomial system $[\mathbb{P}, \mathbb{Q}]$, $\mathbb{P} = 0, \mathbb{Q} \neq 0$ mean the system of polynomial equations $P = 0$ for all $P \in \mathbb{P}$ and inequations $Q \neq 0$ for all $Q \in \mathbb{Q}$.

E11. `dTriSer`

Inputted a list $\mathtt{X} = [t, x_1, \ldots, x_n]$ of variables and an ordinary differential polynomial set $\mathtt{P}$ or system $[\mathtt{P}, \mathtt{Q}]$ in $\mathtt{X}$ and $d^k x_i / d t^k$, $\mathtt{dTriSer}(\mathtt{P}, \mathtt{X})$ or $\mathtt{dTriSer}([\mathtt{P}, \mathtt{Q}], \mathtt{X})$ returns a *differential triangular series* [17] $[\mathbb{T}_1, \mathbb{U}_1], \ldots, [\mathbb{T}_e, \mathbb{U}_e]$ of $\mathtt{P}$ or $[\mathtt{P}, \mathtt{Q}]$ under the variable ordering $x_1 \prec \cdots \prec x_n$.

## charsets

C1. `cfactor`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables with $x_1 \prec \cdots \prec x_n$, an irreducible ascending set $\mathtt{A} = [A_1, \ldots, A_r]$ with $\mathrm{cls}(A_i) = p_i$, and a polynomial $\mathtt{p}$ in $\mathtt{X}$ with $\mathrm{cls}(\mathtt{p}) = p > p_r$ and $\mathtt{prem}(\mathrm{ini}(\mathtt{p}), \mathtt{A}, \mathtt{X}) \neq 0$, $\mathtt{cfactor}(\mathtt{p}, \mathtt{A}, \mathtt{X})$ returns an irreducible factorization of $\mathtt{p}$ over the (algebraic) extension field $Q(u_1, \ldots, u_d, x_1, \ldots, x_{p-1})$, where $x_{p_1}, \ldots, x_{p_r}$ are algebraic elements with each $A_i$ as minimal polynomial for $x_{p_i}$ and all the other variables are transcendental elements.

C2. `charser`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set $\mathtt{P}$ in $\mathtt{X}$, $\mathtt{charser}(\mathtt{P}, \mathtt{X})$ or $\mathtt{charser}(\mathtt{P}, \mathtt{X}, \mathtt{m})$ returns a (*weak*-) *characteristic series* [5] $\mathbb{C}_1, \ldots, \mathbb{C}_e$ of $\mathtt{P}$ under the variable ordering $x_1 \prec \cdots \prec x_n$.

The optional argument `m` may take one of the following five names: `basset`, `wbasset`, `charsetn`, `wcharsetn`, and `trisetc`, of which `charsetn` is the default. A characteristic series of `P` is computed if `m` is `basset`, `charsetn`, or `trisetc`; and a weak-characteristic series of `P` is computed if `m` is one of the others.

C3. `charset`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set $\mathtt{P}$ in $\mathtt{X}$, $\mathtt{charset}(\mathtt{P}, \mathtt{X})$ or $\mathtt{charset}(\mathtt{P}, \mathtt{X}, \mathtt{m})$ returns a (*quasi-, weak-*) *characteristic set* [2] $\mathbb{C}$ of $\mathtt{P}$ with respect to the variable ordering $x_1 \prec \cdots \prec x_n$.

The argument `m` is optional and may take one of the following eight names: `basset`, `wbasset`, `qbasset`, `charsetn`, `wcharsetn`, `qcharsetn`, `trisetc`, and `triset`. If `m` is one of `basset`, `charsetn`, and `trisetc`, then $\mathbb{C}$ is a characteristic set; if `m` is `wbasset` or `wcharsetn`, then $\mathbb{C}$ is a weak-characteristic set; if `m` is one of `qbasset`, `qcharsetn`, and `triset`, then $\mathbb{C}$ is a quasi-characteristic set. If `m` is not given, then the default `charsetn` is used.

C4. `csolve`

Inputted a list or a set $\mathtt{X} = \{x_1, \ldots, x_n\}$ of variables and a polynomial set $\mathtt{P}$ in $\mathtt{X}$, $\mathtt{csolve}(\mathtt{P},\mathtt{X})$ returns (all) the solutions of $\mathtt{P} = 0$ for $x_1, \ldots, x_n$.

C5. `ecs`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables, a polynomial set $\mathtt{P}$ and a nonzero polynomial $\mathtt{p}$ in $\mathtt{X}$, $\mathtt{ecs}([\mathtt{P},\mathtt{p}],\mathtt{X})$ or $\mathtt{ecs}([\mathtt{P},\mathtt{p}],\mathtt{X},\mathtt{m})$ returns an *extended* (*weak*-) *characteristic series* [6] $[\mathbb{C}_1, D_1], \ldots, [\mathbb{C}_e, D_e]$ of $[\mathtt{P}, \{\mathtt{p}\}]$ under the ordering $x_1 \prec \cdots \prec x_n$.

If the optional argument `m` takes one of `basset`, `charsetn`, and `trisetc`, then an extended characteristic series of `P` is computed; if `m` is `wbasset` or `wcharsetn`, then an extended weak-characteristic series of `P` is computed. The default of `m` is `charsetn`.

C6. `eics`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables, a polynomial set $\mathtt{P}$ and a nonzero polynomial $\mathtt{p}$ in $\mathtt{X}$, $\mathtt{eics}([\mathtt{P},\mathtt{p}],\mathtt{X})$ or $\mathtt{eics}([\mathtt{P},\mathtt{p}],\mathtt{X},\mathtt{m})$ returns an (*extended* [6]) *irreducible characteristic series* [10] $[\mathbb{C}_1, D_1], \ldots, [\mathbb{C}_e, D_e]$ of $[\mathtt{P}, \{\mathtt{p}\}]$ under the ordering $x_1 \prec \cdots \prec x_n$.

The optional argument `m` may take one of the following three names: `basset`, `charsetn`, and `trisetc`, with `charsetn` as default.

C7. `ics`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set $\mathtt{P}$ in $\mathtt{X}$, $\mathtt{ics}(\mathtt{P},\mathtt{X})$ or $\mathtt{ics}(\mathtt{P},\mathtt{X},\mathtt{m})$ returns an *irreducible characteristic series* [10] $\mathbb{C}_1, \ldots, \mathbb{C}_e$ of $\mathtt{P}$ under the ordering $x_1 \prec \cdots \prec x_n$.

The optional argument `m` may take `basset`, `charsetn`, or `trisetc`, with `charsetn` as default.

C8. `iniset`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a (quasi-, weak-) ascending set $\mathtt{A}$ with respect to the variable ordering $x_1 \prec \cdots \prec x_n$, $\mathtt{iniset}(\mathtt{A},\mathtt{X})$ returns the set of all distinct (irreducible) factors of the polynomials in $\{\mathrm{ini}(A) \mid A \in \mathtt{A}\}$.

C9. `ivd`

Inputted a set or a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set $\mathtt{P}$ in $\mathtt{X}$, $\mathtt{ivd}(\mathtt{P},\mathtt{X})$ or $\mathtt{ivd}(\mathtt{P},\mathtt{X},\mathtt{m})$ returns a sequence of irredundant polynomial sets $\mathbb{P}_1, \ldots, \mathbb{P}_e$ such that (1) holds and each $\mathrm{Zero}(\mathbb{P}_i)$ as an algebraic variety is *irreducible*. [11]

The optional argument `m` may take one of the following three names: `basset`, `charsetn`, and `trisetc`, with `charsetn` as default.

Except in the case $e = 1$, the above $\mathbb{P}_i$ are all Gröbner bases with respect to the purely lexicographical term order determined by $x_1 \prec \cdots \prec x_n$.

C10. `mcharset`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set $\mathtt{P}$ in $\mathtt{X}$, $\mathtt{mcharset}\ (\mathtt{P},\mathtt{X})$ or $\mathtt{mcharset}(\mathtt{P},\mathtt{X},\mathtt{m})$ returns a *modified* (*quasi-, weak-*) *characteristic set* [3] $\mathbb{C}$ of $\mathtt{P}$ with respect to the variable ordering $x_1 \prec \cdots \prec x_n$.

If the optional argument m is one of basset, charsetn, and trisetc, then $\mathbb{C}$ is a modified characteristic set; if m is wbasset or wcharsetn, then $\mathbb{C}$ is a modified weak-characteristic set; if m is one of qbasset, qcharsetn, and triset, then $\mathbb{C}$ is a modified quasi-characteristic set. If m is not given, then the default charsetn is used.

Function CharSet in E1 is identical to mcharset with m = qcharsetn.

C11. mcs

Inputted a list $X = [x_1, \ldots, x_n]$ of variables and a polynomial set P in X, mcs(P,X), or mcs(P,X,m) returns a (*weak-*) *characteristic series* [5] $\mathbb{C}_1, \ldots, \mathbb{C}_e$ of P under the variable ordering $x_1 \prec \cdots \prec x_n$.

The optional argument m may take one of the following five names: basset, wbasset, charsetn, wcharsetn, and trisetc, of which charsetn is the default. A characteristic series of P is computed if m is basset, charsetn, or trisetc; and a weak-characteristic series of P is computed if m is one of the others.

C12. mecs

Inputted a list $X = [x_1, \ldots, x_n]$ of variables, a polynomial set P and a nonzero polynomial p in X, mecs([P,p],X), or mecs([P,p],X,m) returns an *extended* (*weak-*) *characteristic series* [6] $[\mathbb{C}_1, D_1]$, $\ldots, [\mathbb{C}_e, D_e]$ of $[P, \{p\}]$ under the ordering $x_1 \prec \cdots \prec x_n$.

If the optional argument m takes one of basset, charsetn, and trisetc, then an extended characteristic series of P is computed; if m is wbasset or wcharsetn, then an extended weak-characteristic series of P is computed. The default of m is charsetn.

C13. pid

Inputted a set or a list $X = [x_1, \ldots, x_n]$ of variables and a polynomial set P in X, pid(P,X) or pid(P,X,m) returns a sequence of pairs $[\mathbb{P}_1, \mathbb{F}_1], \ldots, [\mathbb{P}_e, \mathbb{F}_e]$ of polynomial sets such that (2) holds and each $\mathrm{Ideal}(\mathbb{P}_i)$ is a *primary ideal* [12] with $\mathrm{Ideal}(\mathbb{F}_i)$ as its associated prime.

The call pid(P,X) is identical to epsilon[PID](P,X).

The optional argument m may take basset, charsetn, or trisetc, with charsetn as default.

The above $\mathbb{P}_i$ are all Gröbner bases with respect to the purely lexicographical term order determined by $x_1 \prec \cdots \prec x_n$.

C14. qics

Inputted a list $X = [x_1, \ldots, x_n]$ of variables and a polynomial set P in X, qics(P,X) or qics(P,X,m) returns a *quasi-irreducible* [9] (*weak-*) *characteristic series* [5] $\mathbb{C}_1, \ldots, \mathbb{C}_e$ of P under the ordering $x_1 \prec \cdots \prec x_n$.

A quasi-irreducible weak-characteristic series is computed if the optional argument m takes wbasset or wcharsetn, and a quasi-irreducible characteristic series is computed if m takes one of basset, charsetn, and trisetc. The default of m is charsetn.

C15. remset

Inputted a list $X = [x_1, \ldots, x_n]$ of variables, a polynomial p or a polynomial set P in X, and a (quasi-, weak-) ascending set A with respect to the ordering $x_1 \prec \cdots \prec x_n$, remset(p,A,X) returns the *pseudo-remainder* [1] of p with respect to A, and remset(P,A,X) returns the set of all nonzero *pseudo-remainders* [1] of the polynomials in P with respect to A.

## **dcharsets**

D1. dcharset

Inputted a list $X = [t, x_1, \ldots, x_n]$ of variables and a differential polynomial set P in X, dcharset(P,X) or dcharset(P,X,m) returns a *differential* (*quasi-, weak-*) *characteristic set* [15] $\mathbb{C}$ of P with respect to the variable ordering $x_1 \prec \cdots \prec x_n$.

If the optional argument m takes basset or charsetn, then $\mathbb{C}$ is a differential characteristic set; if m takes wbasset or wcharsetn, then $\mathbb{C}$ is a differential weak-characteristic set; if m takes qbasset or qcharsetn, then $\mathbb{C}$ is a differential quasi-characteristic set. When m is not given, the default charsetn is used.

D2. dcs

Inputted a list $X = [t, x_1, \ldots, x_n]$ of variables and a differential polynomial set P or system [P,Q] in X, dcs(P,X) or dcs(P,X,m) returns a *differential* (*weak-*) *characteristic series* [18] $\mathbb{C}_1, \ldots, \mathbb{C}_e$ of P, and dcs([P,Q],X) or dcs([P,Q],X,m) returns a *differential* (*weak-*) *characteristic series* [19] $[\mathbb{C}_1, \mathbb{D}_1], \ldots, [\mathbb{C}_e, \mathbb{D}_e]$ of [P,Q], all under the variable ordering $x_1 \prec \cdots \prec x_n$.

The optional argument m may take one of the four names: basset, wbasset, charsetn, and wcharsetn, of which charsetn is the default. A differential characteristic series is returned if m takes basset or charsetn, and a differential weak-characteristic series is returned if m takes one of the others.

D3. depend

Inputted a list $X = [t, x_1, \ldots, x_n]$ of variables, depend(X) declares that $x_1, \ldots, x_n$ all depend on $t$.

This declaration should be made before any actual computation.

D4. df

Inputted a differential polynomial p, df(p,j) returns the jth order derivative of p with respect to the derivation variable (declared by depend).

If the optional argument j is not given, the default $j = 1$ is used.

D5. dics

Inputted a list $X = [t, x_1, \ldots, x_n]$ of variables and a differential polynomial set P or system [P,Q] in X, dics(P,X) or dics(P,X,m) returns an *irreducible* [21] *differential characteristic series* [18] $\mathbb{C}_1, \ldots, \mathbb{C}_e$ of P, and dics([P,Q],X) or dics([P,Q], X,m) returns an *irreducible* [21] *differential characteristic series* [19] $[\mathbb{C}_1, \mathbb{D}_1], \ldots, [\mathbb{C}_e, \mathbb{D}_e]$ of [P,Q], all under the variable ordering $x_1 \prec \cdots \prec x_n$.

The optional argument m may take basset or charsetn, with the latter as default.

D6. diss

Inputted a list $X = [t, x_1, \ldots, x_n]$ of variables and a differential (quasi-, weak-) ascending set A with respect to $x_1 \prec \cdots \prec x_n$, diss(A,X) returns the set of (all distinct factors of) the initials and separants of the differential polynomials in A.

D7. dmcharset

Inputted a list $X = [t, x_1, \ldots, x_n]$ of variables and a differential polynomial set P in X, dmcharset (P,X) or dmcharset(P,X,m) returns a *modified differential* (*quasi-, weak-*) *characteristic set* [16] $\mathbb{C}$ of P with respect to the variable ordering $x_1 \prec \cdots \prec x_n$.

If the optional argument m takes basset or charsetn, then $\mathbb{C}$ is a modified differential characteristic set; if m takes wbasset or wcharsetn, then $\mathbb{C}$ is a modified differential weak-characteristic set; if m takes qbasset or qcharsetn, then $\mathbb{C}$ is a modified differential quasi-characteristic set. When m is not given, the default charsetn is used.

D8. dmcs

Inputted a list $X = [t, x_1, \ldots, x_n]$ of variables and a differential polynomial set P or system [P,Q] in X, dmcs(P,X) or dmcs(P,X,m) returns a *differential* (*weak-*) *characteristic series* [18] $\mathbb{C}_1, \ldots, \mathbb{C}_e$ of P, and dmcs([P,Q],X) or dmcs([P,Q],X,m) returns a *differential* (*weak-*) *characteristic series* [19] $[\mathbb{C}_1, \mathbb{D}_1], \ldots, [\mathbb{C}_e, \mathbb{D}_e]$ of [P,Q], all under the variable ordering $x_1 \prec \cdots \prec x_n$.

The optional argument m may take one of the four names: basset, wbasset, charsetn, and wcharsetn, of which charsetn is the default. A differential characteristic series is returned if m takes basset or charsetn, and a differential weak-characteristic series is returned if m takes one of the others.

D9. dqics

Inputted a list $X = [t, x_1, \ldots, x_n]$ of variables and a differential polynomial set P or system [P,Q] in X, dqics(P,X) or dqics(P,X,m) returns a *quasi-irreducible* [20] *differential* (*weak-*) *characteristic series* [18] $\mathbb{C}_1, \ldots, \mathbb{C}_e$ of P, and dqics([P,Q],X) or dqics([P,Q],X,m) returns a *quasi-irreducible* [20] *differential* (*weak-*) *characteristic series* [19] $[\mathbb{C}_1, \mathbb{D}_1], \ldots, [\mathbb{C}_e, \mathbb{D}_e]$ of [P,Q], all with respect to the variable ordering $x_1 \prec \cdots \prec x_n$.

A quasi-irreducible differential weak-characteristic series is returned if the optional argument m takes wbasset or wcharsetn, and a quasi-irreducible differential characteristic series is returned if m takes basset or charsetn. The default of m is charsetn.

D10. drs

Inputted a list $X = [t, x_1, \ldots, x_n]$ of variables, a differential polynomial p or set P in X, and a differential (quasi-, weak-) ascending set A under $x_1 \prec \cdots \prec x_n$, drs(p,A,X) returns the *differential pseudo-remainder* [14] of p with respect to A, and drs(P,A,X) returns the set of all nonzero *differential pseudo-remainders* [14] of the differential polynomials in P with respect to A.

D11. format

Inputted a differential polynomial p involving primes in Maple format, format(p,'input') transforms p into the input format with df, and format(p,'latex') transforms p into the LaTeX format.

## trisys

T1. `its`

This function is identical to `ICS` in E2.

T2. `ivd`

This function is identical to `IVD` in E3.

T3. `qits`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set $\mathtt{P}$ or system $[\mathtt{P}, \mathtt{Q}]$ in $\mathtt{X}$, $\mathtt{qits}(\mathtt{P}, \mathtt{X})$ or $\mathtt{qits}([\mathtt{P}, \mathtt{Q}], \mathtt{X})$ returns a *fine* (*quasi-irreducible* [9]) *triangular series* [4] $[\mathbb{T}_1, \mathbb{U}_1], \ldots, [\mathbb{T}_e, \mathbb{U}_e]$ of $\mathtt{P}$ or $[\mathtt{P}, \mathtt{Q}]$ under the variable ordering $x_1 \prec \cdots \prec x_n$.

Function `qits` is identical to `TriSer` in E9.

T4. `rim`

This function is identical to `RIM` in E7.

T5. `triser`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set $\mathtt{P}$ or system $[\mathtt{P}, \mathtt{Q}]$ in $\mathtt{X}$, $\mathtt{triser}(\mathtt{P}, \mathtt{X})$ or $\mathtt{triser}([\mathtt{P}, \mathtt{Q}], \mathtt{X})$ returns a *fine triangular series* [4] $[\mathbb{T}_1, \mathbb{U}_1], \ldots, [\mathbb{T}_e, \mathbb{U}_e]$ of $\mathtt{P}$ or $[\mathtt{P}, \mathtt{Q}]$, and $\mathtt{triser}(\mathtt{P}, \mathtt{X}, x_k)$ or $\mathtt{triser}([\mathtt{P}, \mathtt{Q}], \mathtt{X}, x_k)$ returns a *fine triangular series* [4] $[\mathbb{T}_1, \mathbb{U}_1], \ldots, [\mathbb{T}_e, \mathbb{U}_e]$ of $\mathtt{P}$ or $[\mathtt{P}, \mathtt{Q}]$ with each $[\mathbb{T}_i, \mathbb{U}_i]$ possessing the *projection property* [13] of dimension $k$, all under the variable ordering $x_1 \prec \cdots \prec x_n$.

The dimension is 0 (i.e., the case of *full projection*) if $x_k$ is a variable not in $\{x_1, \ldots, x_n\}$.

T6. `tsolve`

This function is identical to `Tsolve` in E10.

## dtrisys

Δ1. `depend`

Inputted a list $\mathtt{X} = [t, x_1, \ldots, x_n]$ of variables, $\mathtt{depend}(\mathtt{X})$ declares that $x_1, \ldots, x_n$ all depend on $t$.

This declaration should be made before any actual computation.

Δ2. `df`

Inputted a differential polynomial $\mathtt{p}$, $\mathtt{df}(\mathtt{p}, \mathtt{j})$ returns the $\mathtt{j}$th order derivative of $\mathtt{p}$ with respect to the derivation variable (declared by `depend`).

If the optional argument $\mathtt{j}$ is not given, the default $\mathtt{j} = 1$ is used.

Δ3. `dits`

Inputted a list $\mathtt{X} = [t, x_1, \ldots, x_n]$ of variables and a differential polynomial set $\mathtt{P}$ or system $[\mathtt{P}, \mathtt{Q}]$

in X, dits(P,X) or dits([P,Q],X) returns a sequence of irreducible differential triangular sets $\mathbb{T}_1,\ldots,\mathbb{T}_e$ with respect to the variable ordering $x_1 \prec \cdots \prec x_n$, such that

$$\text{d-Zero}(\mathtt{P}/\mathtt{Q}) = \bigcup_{i=1}^{e} \text{d-Zero}(\mathbb{T}_i/\mathtt{diss}(\mathbb{T}_i)\cup\mathtt{Q})$$

(in which Q should be replaced by $\emptyset$ if dits(P,X) is called).

Δ4. dtriser

Inputted a list $\mathtt{X} = [t, x_1, \ldots, x_n]$ of variables and a differential polynomial set P or system [P,Q] in X, dtriser(P,X) or dtriser([P,Q],X) returns a *differential triangular series*[17] $[\mathbb{T}_1,\mathbb{U}_1],\ldots,[\mathbb{T}_e,\mathbb{U}_e]$ of P or [P,Q], and dtriser(P,X,$x_k$) or dtriser([P,Q],X,$x_k$) returns a *differential triangular series*[17] $[\mathbb{T}_1,\mathbb{U}_1],\ldots,[\mathbb{T}_e,\mathbb{U}_e]$ of P or [P,Q] with each $[\mathbb{T}_i,\mathbb{U}_i]$ possessing the *projection property*[22] of dimension $k$, all under the variable ordering $x_1 \prec \cdots \prec x_n$.

Δ5. dqits

Inputted a list $\mathtt{X} = [t, x_1, \ldots, x_n]$ of variables and a differential polynomial set P or system [P,Q] in X, dqits(P,X) or dqits([P,Q],X) returns a (*quasi-irreducible*[20]) *differential triangular series*[17] $[\mathbb{T}_1,\mathbb{U}_1],\ldots,[\mathbb{T}_e,\mathbb{U}_e]$ of P or [P,Q] under the variable ordering $x_1 \prec \cdots \prec x_n$.

Function dqits is identical to dTriSer in E11.

Δ6. drim

Inputted a list $\mathtt{X} = [t, x_1, \ldots, x_n]$ of variables, a differential polynomial p and a differential polynomial set P in X, drim(p,P,X) returns true if p belongs to the radical of the differential ideal generated by the differential polynomials in P, or false otherwise.

## sisys

S1. its

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set P or system [P,Q] in X, its(P,X) or its([P,Q],X) returns an *irreducible triangular series*[10] $\mathbb{T}_1,\ldots,\mathbb{T}_e$ of P or [P,Q] under the variable ordering $x_1 \prec \cdots \prec x_n$.

S2. ivd

Inputted a set or a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set P in X, ivd(P,X) returns a sequence of irredundant polynomial sets $\mathbb{P}_1,\ldots,\mathbb{P}_e$ such that (1) holds and each $\text{Zero}(\mathbb{P}_i)$ as an algebraic variety is *irreducible*.[11]

Except in the case $e = 1$, the above $\mathbb{P}_i$ are all Gröbner bases with respect to the purely lexicographical term order determined by $x_1 \prec \cdots \prec x_n$.

S3. qits

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set P or system [P,Q] in X, qits(P,X) or qits([P,Q],X) returns a *fine* (*quasi-irreducible*[9]) *triangular series*[4] $[\mathbb{T}_1,\mathbb{U}_1],\ldots,[\mathbb{T}_e,\mathbb{U}_e]$ of P or [P,Q] under the ordering $x_1 \prec \cdots \prec x_n$.

S4. `regser`

This function is identical to `RegSer` in E6.

S5. `simser`

This function is identical to `SimSer` in E8.

S6. `ssolve`

Inputted a list or a set $\mathtt{X} = \{x_1, \ldots, x_n\}$ of variables and a polynomial set $\mathtt{P}$ or system $[\mathtt{P}, \mathtt{Q}]$ in $\mathtt{X}$, $\mathtt{ssolve}(\mathtt{P}, \mathtt{X})$ or $\mathtt{ssolve}([\mathtt{P}, \mathtt{Q}], \mathtt{X})$ returns (all) the solutions of $\mathtt{P} = 0$ or $\mathtt{P} = 0, \mathtt{Q} \neq 0$ for $x_1, \ldots, x_n$.

S7. `triser`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set $\mathtt{P}$ or system $[\mathtt{P}, \mathtt{Q}]$ in $\mathtt{X}$, $\mathtt{triser}(\mathtt{P}, \mathtt{X})$ or $\mathtt{triser}([\mathtt{P}, \mathtt{Q}], \mathtt{X})$ returns a *fine triangular series* [4] $[\mathbb{T}_1, \mathbb{U}_1], \ldots, [\mathbb{T}_e, \mathbb{U}_e]$ of $\mathtt{P}$ or $[\mathtt{P}, \mathtt{Q}]$ under the variable ordering $x_1 \prec \cdots \prec x_n$.

S8. `uvd`

Inputted a set or a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial set $\mathtt{P}$ in $\mathtt{X}$, $\mathtt{uvd}(\mathtt{P}, \mathtt{X})$ returns a sequence of polynomial sets $\mathbb{P}_1, \ldots, \mathbb{P}_e$ such that (1) holds and all the irreducible irredundant subvarieties of $\mathrm{Zero}(\mathbb{P}_i)$ are equidimensional for each $i$. Moreover, $\mathrm{Ideal}(\mathbb{P}_i)$ is radical for all $1 \leq i \leq e$.

The above $\mathbb{P}_i$ are all Gröbner bases with respect to the purely lexicographical term order determined by $x_1 \prec \cdots \prec x_n$.

## geother

G1. `Algebraic`

Inputted the predicate specification `T` of a geometric theorem, or a geometric predicate `T` with arguments, or a sequence `T` of points occurring in the current theorem loaded to the GEOTHER session, `Algebraic(T)` translates `T` into an algebraic specification of the theorem or into one or several algebraic expressions, or prints out the coordinates of the points in `T`.

G2. `Chinese`

Inputted the predicate specification `T` of a geometric theorem or a geometric predicate `T` with arguments, `Chinese(T)` translates `T` into a Chinese statement.

G3. `Click`

`Click()` starts a menu-driven GEOTHER user interface.

G4. `Coordinate`

Inputted the predicate specification `T` of a geometric theorem, `Coordinate(T)` reassigns the coordinates of the points appearing in `T` and returns a (new) specification of the same theorem, in which only the third argument (i.e., the list of variables) is modified.

G5. `Demo`

`Demo()` gives an automated demonstration of GEOTHER.

During the demonstration, GEOTHER commands appear automatically and the user only needs to enter semicolon `;` and then return.

G6. `Dprover`

Inputted the algebraic specification `T` of a (differential geometry) theorem in the local theory of curves, `Dprover(T)` proves or disproves `T`, with algebraic subsidiary conditions provided.

G7. `English`

Inputted the predicate specification `T` of a geometric theorem or a geometric predicate `T` with arguments, `English(T)` translates `T` into an English statement.

G8. `Generic`

Inputted the predicate specification `T` of a geometric theorem and a (simple) polynomial `p` (or a set `p` of polynomials) in the coordinates of the points occurring in `T`, `Generic(T, p)` translates the algebraic condition `p <> 0` with respect to `T` into geometric/predicate form.

G9. `Geometric`

Inputted the predicate specification `T` of a geometric theorem, `Geometric(T)` generates one or several diagrams for `T`.

G10. `Gprover`

Inputted the specification `T` of a geometric theorem, `Gprover(T)` or `Gprover(T, Kapur)` proves `T` (without providing subsidiary conditions) or reports that it cannot confirm the theorem.

G11. `GCprover`

Inputted the specification `T` of a geometric theorem, `GCprover(T)` proves `T`, with subsidiary conditions provided, or reports that it cannot confirm the theorem.

G12. `Let`

Inputted a sequence `T` of equations $P_1 = [x_1, y_1], \ldots, P_n = [x_n, y_n]$, `Let(T)` assigns the coordinates $(x_i, y_i)$ to each point $P_i$ for $1 \leq i \leq n$.

G13. `Load`

Inputted the name `T` of a geometric theorem, `Load(T)` reads the specification of the theorem from the built-in library to the GEOTHER session.

G14. `Logic`

Inputted the specification `T` of a geometric theorem, `Logic(T)` translates `T` into a logic formula.

G15. `Print`

Inputted the current geometric theorem `T` in the session and optionally its name `'T'`, `Print(T, 'T')` generates an HTML file (with Java applet) documenting the last manipulations and proof of `T` and invokes Netscape to view the file. If the optional third argument is given as `LaTeX`, then `Print(T, 'T', LaTeX)` generates a PostScript file documenting the theorem and invokes Ghostview to view the file.

G16. `Remark`

Inputted a sequence of strings `s1`, `s2`, ..., `Remark(s1, s2, ...)` stores the remarks `s1`, `s2`, ... into an internal variable. `Remark()` prints out the remarks stored in the internal variable.

G17. `Save`

Inputted a predicate or algebraic specification `T` of a geometric theorem and a string `S`, `Save(T,S)` stores the specification `T` in the current Maple session into the GEOTHER database under the name `S`.

G18. `Search`

Inputted a string `S`, `Search(S)` lists all the names of theorems containing `S` in the library.

If `S` is given as `all` or `All`, then all the theorems in the library are listed.

G19. `Tprover`

Inputted the specification `T` of a geometric theorem, `Tprover(T)` proves or disproves `T`, with subsidiary conditions provided.

G20. `Wprover`

This function is identical to `Prove` in E5.

## miscel

M1. `bezres`

Inputted a list `X` of two variables and a set `P` of three polynomials in `X`, `bezres(P,X)` returns $0$ (when the *Bézout matrix* is singular) or the *Bézout resultant* of `P` with respect to `X`.

M2. `implicit`

Inputted an estimated degree `n`, a list `X` of two variables, and a set `P` of three polynomials $P - xA, Q - yB, R - zC$, where $P, Q, R, A, B, C$ are polynomials in `X`, `implicit(P,X,n)` returns the implicit polynomial $F$ in $x, y, z$ if the rational surface defined by $\mathtt{P} = 0$ has an implicit equation $F = 0$ of total degree less than or equal to `n`, or $0$ otherwise. If the dependency of the two parameters is detected, then the constant $1$ is returned.

M3. `indext`

Inputted a list $\mathtt{X} = [x_1, \ldots, x_n]$ of variables and a polynomial `P` or a list `P` of polynomials in `X`, `indext(P,X)` returns the index triple of `P` or the list of index triples of the polynomials in `P` with respect to $x_1 \prec \cdots \prec x_n$.

M4. `licon`

Inputted an odd integer $\mathtt{m} > 2$, an odd integer $\mathtt{n} > \mathtt{m}$, a polynomial `P` of the form $y + P(x,y)$, and a polynomial `Q` of the form $-x + Q(x,y)$, where $P$ and $Q$ are polynomials in $x$ and $y$, with terms of total degree $> 1$, `licon(P,Q,m,n)` returns a list $[v_{\mathtt{m}}, v_{\mathtt{m}+2}, \ldots, v_{\mathtt{n}}]$ of polynomials in the coefficients of $P$ and $Q$, called the *Liapunov constants* of the differential system

$$\frac{dx}{dt} = y + P(x,y), \quad \frac{dy}{dt} = -x + Q(x,y).$$

M5. `macres`

Inputted a list X of $n$ variables and a set P of $n$ homogeneous polynomials in X, $\texttt{macres}(\texttt{P},\texttt{X})$ returns the *Macaulay resultant* of P with respect to X.

M6. `normat`

Inputted a list $\texttt{X} = [x_1,\ldots,x_n]$ of variables and a triangular set T in X, $\texttt{normat}(\texttt{T},\texttt{X})$ returns a sequence of *normal* [23] triangular sets $\mathbb{T}_1,\ldots,\mathbb{T}_e$ under the variable ordering $x_1 \prec \cdots \prec x_n$, such that

$$\mathrm{Zero}(\mathbb{T}/\texttt{iniset}(\mathbb{T})) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{T}_i/\texttt{iniset}(\mathbb{T}_i) \cup \texttt{iniset}(\mathbb{T})).$$

M7. `subres`

Inputted a variable x and two polynomials p and q in x with $\texttt{degree}(\texttt{p},\texttt{x}) \geq \texttt{degree}(\texttt{q},\texttt{x}) > 0$, $\texttt{subres}(\texttt{p},\texttt{q},\texttt{x})$ returns the sequence of *subresultants* of p and q with respect to the variable x.

## Definitions

1. The *pseudo-remainder* of a polynomial $P$ with respect to a quasi-ascending set $\mathbb{A} = [A_1,\ldots,A_r]$ with $\mathrm{cls}(A_i) = p_i$ is

$$\texttt{prem}(P,\mathbb{A}) := \begin{cases} 0 & \text{if } r = 1 \text{ and } p_1 = 0, \\ \texttt{prem}(\ldots\texttt{prem}(P,A_r,x_{p_r}),\ldots,A_1,x_{p_1}) & \text{otherwise.} \end{cases}$$

2. A (quasi-, weak-) ascending set $\mathbb{C}$ is called a (*quasi-, weak-*) *characteristic set* of a polynomial set $\mathbb{P}$ if $\mathbb{C} \subset \mathrm{Ideal}(\mathbb{P})$ and $\texttt{remset}(\mathbb{P},\mathbb{C}) = \emptyset$.

3. A (quasi-, weak-) ascending set $\mathbb{C}$ is called an $\mathbb{F}$-*modified* (*quasi-, weak-*) *characteristic set* of a polynomial set $\mathbb{P}$ if $\mathrm{Zero}(\mathbb{P}/\mathbb{F}) \subset \mathrm{Zero}(\mathbb{C})$ and $\texttt{remset}(\mathbb{P},\mathbb{C}) = \emptyset$, where $\mathbb{F}$ is another polynomial set.

4. A sequence $\Psi$ of (fine) triangular systems $[\mathbb{T}_1,\mathbb{U}_1],\ldots,[\mathbb{T}_e,\mathbb{U}_e]$ is called a (*fine*) *triangular series* of a polynomial system $[\mathbb{P},\mathbb{Q}]$ if

$$\mathrm{Zero}(\mathbb{P}/\mathbb{Q}) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{T}_i/\mathbb{U}_i). \tag{3}$$

In the case $\mathbb{Q} = \emptyset$, $\Psi$ is also called a (*fine*) *triangular series* of the polynomial set $\mathbb{P}$.

5. A sequence of (weak-) ascending sets $\mathbb{C}_1,\ldots,\mathbb{C}_e$ is called a (*weak-*) *characteristic series* of a polynomial set $\mathbb{P}$ if

$$\mathrm{Zero}(\mathbb{P}) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{C}_i/\texttt{iniset}(\mathbb{C}_i))$$

and $\texttt{remset}(\mathbb{P},\mathbb{C}_i) = \emptyset$ for all $i$.

6. Let $\mathbb{C}_1, \ldots, \mathbb{C}_e$ be (weak-) ascending sets and $\mathbb{D}_1, \ldots, \mathbb{D}_e$ polynomial sets. The sequence $[\mathbb{C}_1, \mathbb{D}_1]$, $\ldots, [\mathbb{C}_e, \mathbb{D}_e]$ is called an *extended* (*weak-*) *characteristic series* of a polynomial system $[\mathbb{P}, \mathbb{Q}]$ if

$$\mathrm{Zero}(\mathbb{P}/\mathbb{Q}) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{C}_i/\mathbb{D}_i),$$

$[\mathbb{C}_i, \mathbb{D}_i]$ is a fine triangular system and $\mathtt{remset}(\mathbb{P}, \mathbb{C}_i) = \emptyset$ for all $i$.

$[\mathbb{C}_i, D_i]$ may be used instead of $[\mathbb{C}_i, \mathbb{D}_i]$ when $\mathbb{D}_i$ consists of a single polynomial $D_i$.

7. A sequence of regular systems $[\mathbb{T}_1, \mathbb{U}_1], \ldots, [\mathbb{T}_e, \mathbb{U}_e]$ is called a *regular series* of a polynomial system $[\mathbb{P}, \mathbb{Q}]$, or of the polynomial set $\mathbb{P}$ in case $\mathbb{Q} = \emptyset$, if (3) holds.

8. A sequence of simple systems $[\mathbb{T}_1, \tilde{\mathbb{T}}_1], \ldots, [\mathbb{T}_e, \tilde{\mathbb{T}}_e]$ is called a *simple series* of a polynomial system $[\mathbb{P}, \mathbb{Q}]$ (or set $\mathbb{P}$ in case $\mathbb{Q} = \emptyset$) if

$$\mathrm{Zero}(\mathbb{P}/\mathbb{Q}) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{T}_i/\tilde{\mathbb{T}}_i).$$

9. A triangular or (weak-) characteristic series $\Psi$ (of a polynomial set or system) is said to be *quasi-irreducible* if all the polynomials in the triangular or (weak-) ascending sets of $\Psi$ are irreducible over $\mathcal{K}$.

10. A fine triangular or (extended) characteristic series $\Psi$ (of a polynomial set $\mathbb{P}$ or system $[\mathbb{P}, \mathbb{Q}]$) is said to be *irreducible* if all the triangular or ascending sets in $\Psi$ are irreducible. In this case, (3) implies that

$$\mathrm{Zero}(\mathbb{P}/\mathbb{Q}) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{T}_i/\mathtt{iniset}(\mathbb{T}_i) \cup \mathbb{Q}),$$

and the sequence $\mathbb{T}_1, \ldots, \mathbb{T}_e$ is also called an *irreducible triangular* (or *characteristic*) *series* of $[\mathbb{P}, \mathbb{Q}]$ (or of the polynomial set $\mathbb{P}$ when $\mathbb{Q} = \emptyset$).

11. An algebraic variety $\mathcal{V}$ is said to be *irreducible* if it cannot be expressed as the union of two true subvarieties of $\mathcal{V}$.

12. An ideal $\mathfrak{I}$ is said to be *primary* if $FG \in \mathfrak{I}$ and $F \notin \mathfrak{I}$ imply that there exists an integer $q > 0$ such that $G^q \in \mathfrak{I}$.

13. Let $[\mathbb{T}, \mathbb{U}]$ be a fine triangular system and $k$ a nonnegative integer. $[\mathbb{T}, \mathbb{U}]$ is said to possess the *projection property* of dimension $k$ if

$$\mathrm{Zero}(\mathbb{T} \cap \mathcal{K}[x_1, \ldots, x_i]/\mathbb{U} \cap \mathcal{K}[x_1, \ldots, x_i]) \subset \mathrm{Proj}_{x_1, \ldots, x_i} \mathrm{Zero}(\mathbb{T}/\mathbb{U})$$

holds for $i = k$ and all $i \in \{\mathrm{cls}(T) \mid T \in \mathbb{T}, \mathrm{cls}(T) > k\}$.

14. For any differential polynomial $P$ and differential quasi-ascending set $\mathbb{A} = [A_1, \ldots, A_r]$,

$$\text{d-prem}(P, \mathbb{A}) := \text{d-prem}(\ldots \text{d-prem}(P, A_r), \ldots, A_1)$$

is called the *differential pseudo-remainder* of $P$ with respect to $\mathbb{A}$.

15. A differential (quasi-, weak-) ascending set $\mathbb{C}$ is called a *differential* (*quasi-, weak-*) *characteristic set* of a differential polynomial set $\mathbb{P}$ if $\mathbb{C}$ is contained in the differential ideal generated by the differential polynomials in $\mathbb{P}$ and $\mathtt{drs}(\mathbb{P},\mathbb{C}) = \emptyset$.

16. A differential (quasi-, weak-) ascending set $\mathbb{C}$ is called an $\mathbb{F}$-*modified differential* (*quasi-, weak-*) *characteristic set* of $\mathbb{P}$ if $\text{d-Zero}(\mathbb{P}/\mathbb{F}) \subset \text{d-Zero}(\mathbb{C})$ and $\mathtt{drs}(\mathbb{P},\mathbb{C}) = \emptyset$, where $\mathbb{F}$ is another differential polynomial set.

17. A sequence of (fine) *differential* triangular systems $[\mathbb{T}_1, \mathbb{U}_1], \ldots, [\mathbb{T}_e, \mathbb{U}_e]$ is called a (*fine*) *differential triangular series* of a differential polynomial system $[\mathbb{P}, \mathbb{Q}]$, or of $\mathbb{P}$ in case $\mathbb{Q} = \emptyset$, if

$$\text{d-Zero}(\mathbb{P}/\mathbb{Q}) = \bigcup_{i=1}^{e} \text{d-Zero}(\mathbb{T}_i/\mathbb{U}_i).$$

18. A sequence of differential (weak-) ascending sets $\mathbb{C}_1, \ldots, \mathbb{C}_e$ is called a *differential* (*weak-*) *characteristic series* of a differential polynomial set $\mathbb{P}$ if

$$\text{d-Zero}(\mathbb{P}) = \bigcup_{i=1}^{e} \text{d-Zero}(\mathbb{C}_i/\mathtt{diss}(\mathbb{C}_i))$$

and $\mathtt{drs}(\mathbb{P},\mathbb{C}_i) = \emptyset$ for all $i$.

19. Let $\mathbb{C}_1, \ldots, \mathbb{C}_e$ be differential (weak-) ascending sets and $\mathbb{D}_1, \ldots, \mathbb{D}_e$ differential polynomial sets. The sequence $[\mathbb{C}_1, \mathbb{D}_1], \ldots, [\mathbb{C}_e, \mathbb{D}_e]$ is called a *differential* (*weak-*) *characteristic series* of a differential polynomial system $[\mathbb{P}, \mathbb{Q}]$ if

$$\text{d-Zero}(\mathbb{P}/\mathbb{Q}) = \bigcup_{i=1}^{e} \text{d-Zero}(\mathbb{C}_i/\mathbb{D}_i),$$

$[\mathbb{C}_i, \mathbb{D}_i]$ is a fine differential triangular system and $\mathtt{drs}(\mathbb{P},\mathbb{C}_i) = \emptyset$ for all $i$.

20. A differential triangular or (weak-) characteristic series $\Psi$ is said to be *quasi-irreducible* if all the differential polynomials in the differential triangular or (weak-) ascending sets of $\Psi$, considered as polynomials (with the occurring derivatives as variables), are irreducible over $\mathcal{F}$.

21. A differential characteristic series $\Psi$ is said to be *irreducible* if all the differential ascending sets in $\Psi$ are irreducible.

22. For any nonnegative integer $k$, a fine differential triangular system $\mathfrak{T}$ is said to possess the *projection property* of dimension $k$ if $\mathfrak{T}$ as a fine triangular system with the occurring derivatives as variables possesses the *projection property* of dimension $k$.

23. A triangular set $\mathbb{T} = [T_1, \ldots, T_r]$ is said to be *normal* if $\mathtt{degree}(\text{ini}(T_i), \text{lvar}(T_j)) = 0$ for all $0 \leq j < i \leq r$, i.e., the initial $\text{ini}(T_i)$ does not involve any leading variables $\text{lvar}(T_1), \ldots, \text{lvar}(T_{i-1})$ for $1 < i \leq r$.

## Basic Notions

All polynomials mentioned in this document are in the variables $x_1, \ldots, x_n$ with coefficients in $\mathcal{K}$. The coefficient domain $\mathcal{K}$ may be the field $\mathcal{Q}$ of rational numbers or the ring $\mathcal{Q}[u_1, \ldots, u_d]$, where the $u_i$ are considered as parametric variables. $\tilde{\mathcal{K}}$ denotes an extension field of $\mathcal{K}$. A *polynomial set* is a finite set of nonzero polynomials, and a *polynomial system* is a pair of polynomial sets. Any polynomial in $\mathcal{Q}[u_1, \ldots, u_d]$ is also called a *constant*.

- For any polynomial system $[\mathbb{P}, \mathbb{Q}]$,

  $$\mathrm{Zero}(\mathbb{P}/\mathbb{Q}) := \left\{ (\bar{x}_1, \ldots, \bar{x}_n) \in \bar{\mathcal{K}}^n \;\middle|\; \begin{array}{l} P(\bar{x}_1, \ldots, \bar{x}_n) = 0, \forall P \in \mathbb{P}, \\ Q(\bar{x}_1, \ldots, \bar{x}_n) \neq 0, \forall Q \in \mathbb{Q} \end{array} \right\},$$

  where $\bar{\mathcal{K}}$ is an algebraically closed field containing $\mathcal{K}$. $\mathrm{Zero}(\mathbb{P})$ may be used instead of $\mathrm{Zero}(\mathbb{P}/\mathbb{Q})$ when $\mathbb{Q} \subset \mathcal{K} \setminus \{0\}$.

- For any polynomial set $\mathbb{P}$, $\mathrm{Ideal}(\mathbb{P})$ denotes the ideal generated by all the polynomials in $\mathbb{P}$, and $\sqrt{\mathrm{Ideal}(\mathbb{P})}$ is the radical of $\mathrm{Ideal}(\mathbb{P})$.

- For any nonzero polynomial $P$, the *class* of $P$, denoted by $\mathrm{cls}(P)$, is the biggest index of the variables $x_k$ that effectively occurs in $P$ if $P$ is not a constant, or $0$ otherwise.

- For any nonconstant polynomial $P$ of class $p$, $x_p$ is called the *leading variable* of $P$, denoted by $\mathrm{lvar}(P)$. A polynomial $Q$ is said to be *reduced* with respect to $P$ if $\texttt{degree}(Q, x_p) < \texttt{degree}(P, x_p)$, where $\texttt{degree}(P, x_k)$ denotes the degree of $P$ in $x_k$ as in Maple.

- Let $p = \mathrm{cls}(P) > 0$; the leading coefficient of $P$ with respect to $x_p$ is called the *initial* of $P$, denoted by $\mathrm{ini}(P)$.

- An *index triple* $[t \;\; \mathrm{lvar}(P) \;\; d]$ is used to characterize $P$, where $t$ is the number of terms of $P$ and $d$ the degree of $P$ in $\mathrm{lvar}(P)$.

- A finite nonempty ordered set $[T_1, T_2, \ldots, T_r]$ of nonconstant polynomials is called a *triangular set* if

  $$\mathrm{cls}(T_1) < \mathrm{cls}(T_2) < \cdots < \mathrm{cls}(T_r).$$

  A *quasi-ascending set* is either a triangular set or a set of a single nonzero constant.

- A quasi-ascending set $\mathbb{A} = [A_1, \ldots, A_r]$ is called an *ascending set* if in the case $r > 1$, $A_j$ is reduced with respect to $A_i$ for each pair $j > i$.

  $\mathbb{A}$ is called a *weak-ascending set* if in the case $r > 1$, $\mathrm{ini}(A_j)$ is reduced with respect to $A_i$ for each pair $j > i$.

  A (weak-, quasi-) ascending set is said to be *contradictory* if $r = 1$ and $\mathrm{cls}(A_1) = 0$.

- A polynomial system $[\mathbb{T},\mathbb{U}]$ is called a *triangular system* if $\mathbb{T}$ is a triangular set and $I(\bar{x}_1,\ldots,\bar{x}_k)\neq 0$ for any $I\in\{\mathrm{ini}(T)\mid T\in\mathbb{T}\}$ and $(\bar{x}_1,\ldots,\bar{x}_n)\in \mathrm{Zero}(\mathbb{T}\cap\mathcal{K}[x_1,\ldots,x_k]/\mathbb{U})$, where $k=\mathrm{cls}(I)$.

  A triangular system $[\mathbb{T},\mathbb{U}]$ is said to be *fine* if $\mathtt{prem}(U,\mathbb{T})\neq 0$ for all $U\in\mathbb{U}$. A triangular set $\mathbb{T}$ is said to be *fine* if $[\mathbb{T},\{\mathrm{ini}(T)\mid T\in\mathbb{T}\}]$ is fine.

- A triangular system $[\mathbb{T},\mathbb{U}]$ is called a *regular system* if

  (a) for any $T\in\mathbb{T}$ and $U\in\mathbb{U}$, $\mathrm{cls}(T)\neq\mathrm{cls}(U)$;

  (b) for any $1\le k\le n$, $U\in\mathbb{U}$ of class $k$, and

  $$(\bar{x}_1,\ldots,\bar{x}_{k-1})\in \mathrm{Zero}(\mathbb{T}\cap\mathcal{K}[x_1,\ldots,x_{k-1}]/\mathbb{U}\cap\mathcal{K}[x_1,\ldots,x_{k-1}]),$$

  $\mathrm{ini}(U)(\bar{x}_1,\ldots,\bar{x}_{k-1})\neq 0$.

- A pair $[\mathbb{T},\tilde{\mathbb{T}}]$, in which $\mathbb{T}$ and $\tilde{\mathbb{T}}$ are either triangular sets or the empty set, is called a *simple system* if

  (a) $\mathbb{T}\cap\tilde{\mathbb{T}}=\emptyset$ and $\mathbb{T}\cup\tilde{\mathbb{T}}$ can be reordered as a triangular set;

  (b) for every $T\in\mathbb{T}\cup\tilde{\mathbb{T}}$ of class $k$ and any $(\bar{x}_1,\ldots,\bar{x}_{k-1})\in\mathrm{Zero}(\mathbb{T}\cap\mathcal{K}[x_1,\ldots,x_{k-1}]/\tilde{\mathbb{T}}\cap\mathcal{K}[x_1,\ldots,x_{k-1}])$,

  $$\mathrm{ini}(T)(\bar{x}_1,\ldots,\bar{x}_{k-1})\neq 0 \text{ and } T(\bar{x}_1,\ldots,\bar{x}_{k-1},x_k) \text{ is squarefree.}$$

- A fine triangular set $[T_1,\ldots,T_r]$ is said to be *irreducible* if there do not exist any integer $k$ ($1\le k\le n$) and polynomials $D,F,G$ with $\mathrm{cls}(D)<\mathrm{cls}(F)=\mathrm{cls}(G)=\mathrm{cls}(T_k)$ such that $DT_k=FG$ (when $k=1$) or

  $$\mathtt{prem}(DT_k-FG,[T_1,\ldots,T_{k-1}])=0.$$

- For any polynomial system $\mathfrak{P}$ and $1\le i\le n-1$, the *projection* of $\mathrm{Zero}(\mathfrak{P})$ onto $x_1,\ldots,x_i$ is

  $$\mathrm{Proj}_{x_1,\ldots,x_i}\mathrm{Zero}(\mathfrak{P}):=\left\{(\bar{x}_1,\ldots,\bar{x}_i)\in\tilde{\mathcal{K}}^i \;\middle|\; \begin{array}{c}\exists\bar{x}_{i+1},\ldots,\bar{x}_n\in\tilde{\mathcal{K}} \text{ such that}\\ (\bar{x}_1,\ldots,\bar{x}_n)\in\mathrm{Zero}(\mathfrak{P})\end{array}\right\}.$$

  Moreover,

  $$\mathrm{Proj}_{x_1,\ldots,x_n}\mathrm{Zero}(\mathfrak{P}):=\mathrm{Zero}(\mathfrak{P}),\quad \mathrm{ProjZero}(\mathfrak{P}):=\begin{cases}\emptyset & \text{if } \mathrm{Zero}(\mathfrak{P})=\emptyset,\\ \{0\} & \text{otherwise.}\end{cases}$$

- The set of all zeros of a polynomial set considered as points in the $n$-dimensional affine space with coordinates $x_1,\ldots,x_n$ is called an *algebraic variety*, or *variety* for short.

  A variety $\mathcal{V}_1$ is called a *true* subvariety of another variety $\mathcal{V}_2$ if $\mathcal{V}_1$ is not the trivial empty variety, $\mathcal{V}_1\subset\mathcal{V}_2$, and $\mathcal{V}_1\neq\mathcal{V}_2$.

Let $\mathcal{F} = Q(u_1,\ldots,u_d,t)$ be the field of rational functions of $u_1,\ldots,u_d$ and $t$. A *differential polynomial* is a polynomial in $x_1,\ldots,x_n$ and their derivatives with respect to the derivation variable $t$ with coefficients in $\mathcal{F}$. The set of all such *ordinary* differential polynomials is denoted by $\mathcal{F}\{x_1,\ldots,x_n\}$. A *differential polynomial set* is a finite set of nonzero differential polynomials, and a *differential polynomial system* is a pair of differential polynomial sets.

- For any differential polynomial system $[\mathbb{P},\mathbb{Q}]$, $\text{d-Zero}(\mathbb{P}/\mathbb{Q})$ denotes the set of all common differential zeros (in an algebraically closed differential field containing $\mathcal{F}$) of the differential polynomials in $\mathbb{P}$ which are not differential zeros of any differential polynomial in $\mathbb{Q}$. $\text{d-Zero}(\mathbb{P})$ may be used instead of $\text{d-Zero}(\mathbb{P}/\mathbb{Q})$ when $\mathbb{Q} \subset \mathcal{F} \setminus \{0\}$.

- Differentiation of functions $x_i$ is indicated by means of a second subscript as $x_{ij} = d^j x_i/dt^j$, with $x_{i0} = x_i$. Let $P \in \mathcal{F}\{x_1,\ldots,x_n\}$ be a nonzero differential polynomial. The $j$th order *derivative* of $P$ is obtained by differentiating $P$ $j$ times with respect to $t$, regarding $x_1,\ldots,x_n$ as functions of $t$.

- The greatest $j$, if exists, such that $\texttt{degree}(P,x_{ij}) > 0$ is called the *order* of $P$ with respect to $x_i$, denoted by $\text{ord}(P,x_i)$. If $\texttt{degree}(P,x_{ij}) = 0$ for all $j \geq 0$, then define $\text{ord}(P,x_i) = -1$. Let

$$q = \text{ord}(P,x_i), \quad d = \texttt{degree}(P,x_{iq});$$

  the pair $\langle q,d\rangle$ is called the *rank* of $P$ with respect to $x_i$, denoted by $\text{rank}(P,x_i)$. Order $\langle q,d\rangle \prec \langle q',d'\rangle$ if $q < q'$ or $q = q'$, and $d < d'$. Fix the variable ordering as

$$x_1 \prec x_2 \prec \cdots \prec x_n,$$

  and order $x_{ij} \prec x_{ik}$ if $j < k$.

- For any nonzero differential polynomial $P$, the *class* of $P$, denoted by $\text{cls}(P)$, is the biggest index $p$ such that $\texttt{degree}(P,x_{pj}) > 0$ for some $j$ if $P \notin \mathcal{F}$, or 0 otherwise.

- Let $P$ be a nonzero differential polynomial with

$$\text{cls}(P) = p > 0, \quad \text{rank}(P,x_p) = \langle q,d\rangle,$$

  written as

$$P = P_0 x_{pq}^d + P_1 x_{pq}^{d-1} + \cdots + P_d, \quad P_0 \neq 0,$$

  where $\text{ord}(P_i,x_p) < q$ for each $i$. The variable $x_{pq}$ is called the *lead* of $P$, denoted by $\text{lead}(P)$, and $P_0$ is called the *initial* of $P$, denoted by $\text{ini}(P)$. The differential polynomial $\partial P/\partial x_{pq}$ is called the *separant* of $P$, denoted by $\text{sep}(P)$.

- Let $Q$ be another differential polynomial. Pseudo-dividing $Q$ by $P$ and its derivatives in $x_p$, one may get a differential pseudo-remainder formula of the form

$$\text{sep}(P)^\alpha \text{ini}(P)^\beta Q = H_1 \frac{d^{k_1}P}{dt^{k_1}} + \cdots + H_s \frac{d^{k_s}P}{dt^{k_s}} + R,$$

  where $\alpha,\beta,k_j$ are nonnegative integers and $R$ a differential polynomial with $\text{rank}\,(R,x_p) \prec \langle q,d\rangle$. $R$ is called the *differential pseudo-remainder* of $Q$ with respect to $P$ and denoted by $\text{d-prem}(Q,P)$. Put $\text{d-prem}(Q,P) = 0$ when $P \neq 0$ and $\text{cls}(P) = 0$.

- A finite nonempty ordered set $\mathbb{A} = [A_1, \ldots, A_r]$ of nonzero differential polynomials is called a *differential quasi-ascending set* if either $r = 1$ and $\mathrm{cls}(A_1) = 0$, or $r \geq 1$ and

  $$0 < \mathrm{cls}(A_1) < \mathrm{cls}(A_2) < \cdots < \mathrm{cls}(A_r).$$

  In the latter case (where $\mathrm{cls}(A_1) > 0$), $\mathbb{A}$ is also called a *differential triangular set*.

  A differential quasi-ascending set $\mathbb{A}$ as above is called a *differential weak-ascending set* if

  $$\mathrm{rank}(\mathrm{sep}(A_j), x_{p_i}) \prec \mathrm{rank}(A_i, x_{p_i})$$

  for each pair $j > i$, where $p_i = \mathrm{cls}(A_i)$. It is called a *differential ascending set* if $\mathrm{rank}(A_j, x_{p_i}) \prec \mathrm{rank}(A_i, x_{p_i})$ for each pair $j > i$.

- A differential polynomial system $[\mathbb{T}, \mathbb{U}]$ is called a *differential triangular system* if $\mathbb{T}$ is a differential triangular set and $J(\bar{\boldsymbol{x}}) \neq 0$ for all $J \in \mathtt{diss}(\mathbb{T})$ and $\bar{\boldsymbol{x}} \in \text{d-Zero}(\mathbb{T} \cap \mathcal{F}\{x_1, \ldots, x_k\}/\mathbb{U})$, where $k = \mathrm{cls}(J)$.

  A differential triangular system $[\mathbb{T}, \mathbb{U}]$ is said to be *fine* if $\text{d-prem}(U, \mathbb{T}) \neq 0$ for all $U \in \mathbb{U}$. A differential triangular set $\mathbb{T}$ is said to be *fine* if $[\mathbb{T}, \mathtt{diss}(\mathbb{T})]$ is fine.

- A fine differential triangular set $[T_1, \ldots, T_r]$ is said to be *irreducible* if for every $1 \leq k \leq r$ there do not exist differential polynomials $D$ and $F, G$ with

  $$\mathrm{lead}(D) \prec \mathrm{lead}(T_k), \quad \mathrm{lead}(F) = \mathrm{lead}(G) = \mathrm{lead}(T_k)$$

  such that $DT_k = FG$ (when $k = 1$) or

  $$\text{d-prem}(DT_k - FG, [T_1, \ldots, T_{k-1}]) = 0.$$

The main contents of this document are excerpted from the author's book "Elimination Practice: Software Tools and Applications" published in 2003 by Imperial College Press, London.

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